Linear Algebra
Chapter 1: Matrices

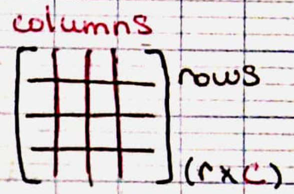
I. Défintions

→ Matrix

It's a table of number (elements), that have a dimension (r.c)

↳ with: r: number of rows

c: number of columns



↳ rows x columns = nb of elements

①

$$A = \begin{bmatrix} 3 & 1 & 2 \\ 4 & -7 & 6 \\ -9 & 2 & 0 \end{bmatrix}_{3\times 3} \Rightarrow A\,(3 \times 3)$$

$3 \times 3 = 9$: this matrix have 9 elements

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}_{3\times 3} \Rightarrow \begin{aligned} a_{12} &= 1 \\ a_{23} &= 6 \\ a_{32} &= 2 \end{aligned}$$

②

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}_{3\times 2}$$

II- Types of matrix

→ **Square matrix**

A square matrix is a matrix that have the same nb of rows and columns (rows = columns) with size $(n \times n)$

diagonal elements

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}_{3\times 3} \qquad A = \begin{bmatrix} 3 & 1 & 2 \\ 2 & 1 & 5 \\ 5 & 6 & 7 \end{bmatrix}_{3\times 3}$$

→ **Horizontal matrix**

A matrix with 1 row and many columns

We call it also a row vector.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \end{bmatrix}_{1\times 4} \qquad A = \begin{bmatrix} 2 & 6 & -4 & 7 \end{bmatrix}_{1\times 4}$$

→ **Vertical matrix**

A matrix with 1 columns and many rows

we callit also a column vector

$$A = \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \\ a_{41} \end{bmatrix}_{4\times 1} \qquad A = \begin{bmatrix} 2 \\ 6 \\ -4 \\ 7 \end{bmatrix}_{4\times 1}$$

→ The zero matrix

A matrix $(m \times n)$ that every element in it is equal zero.

↳ (2×3) zero matrix : $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{2 \times 3}$

$\neq$

↳ (3×2) zero matrix : $\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}_{3 \times 2}$

→ rectangular matrix

A $(m \times n)$ matrix with $m \neq n$.

$A = \begin{bmatrix} 3 & 2 & 1 & 6 \\ 5 & 4 & 7 & 8 \end{bmatrix}_{2 \times 4}$

$A = \begin{bmatrix} 9 & 8 & 1 & 4 \\ 6 & 5 & 3 & 2 \\ 0 & 9 & 4 & 8 \end{bmatrix}_{3 \times 4}$

→ Lower triangular square matrix

A matrix with all of the element above the diagonal equal 0.

$A = \begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix}_{3 \times 3}$

$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 6 & 0 & 0 \\ 3 & 7 & 2 & 0 \\ 4 & 8 & 5 & 9 \end{bmatrix}_{4 \times 4}$

→ Upper triangular square matrix

A $(n \times n)$ square matrix with all of the element under the diagonal equal 0

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}_{3\times3} \qquad A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}_{3\times3}$$

→ Identity matrix (I_n)

A $(n \times n)$ square matrix with the diagonal elements equal 1 and the other elements equal 0

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3\times3} \qquad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}_{4\times4} \qquad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_{2\times2}$$

→ Symmetric matrix

A $(n \times n)$ square matrix with $a_{ij} = a_{ji}$

$$A = \begin{bmatrix} 1 & 3 & 4 \\ 3 & 2 & 5 \\ 4 & 5 & 3 \end{bmatrix}_{3\times3} \implies \begin{aligned} a_{21} &= a_{12} \\ a_{31} &= a_{13} \\ a_{23} &= a_{32} \end{aligned}$$

<u>N.B</u>: in symmetric matrix $A = A^t$

III- Operation in matrix

→ Equality of matrices

. Two equal matrices have the same dimension with every element in the first matrix equal the corresponding element in the second matrix.

. Let A and B two equal matrices:

$$A = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}_{2\times 2} \qquad B = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}_{2\times 2} \Rightarrow$$

A and B have the same dimension (2×2)

$a_1 = b_1$

$a_2 = b_2$

$a_3 = b_3$

$a_4 = b_4$

$$A = \begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix}_{2\times 2} \qquad B = \begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix}_{2\times 2}$$

→ Addition of matrices

. To do the addition of two matrices A and B:

→ A and B should have the same dimension $(n\times m)$

→ $A + B$ give the matrix C with $(n\times m)$ dimension

→ $a_{11} + b_{11} = c_{11}$; $a_{12} + b_{12} = c_{12}$ ...

$3+4=7$

$$\underset{A}{\begin{bmatrix} 3 & -1 \\ 2 & -4 \end{bmatrix}_{2\times 2}} + \underset{B}{\begin{bmatrix} 4 & -1 \\ 3 & 7 \end{bmatrix}_{2\times 2}} = \underset{C}{\begin{bmatrix} 7 & -2 \\ 5 & 3 \end{bmatrix}_{2\times 2}}$$

Properties of Addition:

- Let A, B, and C be matrices then the following properties:

1. Commutative law of addition: $A + B = B + A$
2. Associative law of addition: $(A + B) + C = A + (B + C)$
3. Existence of an additive identity: $A + 0 = A$
4. Existence of additive inverse: $A + (-A) = 0$

Substraction of matrices

. To do substraction of two matrices A and B:

- A and B should have same dimension $(n \times m)$
- $A - B$ give the matrix C with $(n \times m)$ dimension
- $a_{11} - b_{11} = c_{11}$; $a_{12} - b_{12} = c_{12}$ ...
- $A - B \neq B - A$

$$A = \begin{bmatrix} 3 & -1 \\ 2 & -4 \end{bmatrix}_{2 \times 2} \quad \text{and} \quad B = \begin{bmatrix} 4 & -1 \\ 3 & 7 \end{bmatrix}_{2 \times 2}$$

$$A - B = \overset{C}{\begin{bmatrix} -1 & 0 \\ -1 & -11 \end{bmatrix}}_{2 \times 2} \qquad B - A = \overset{C}{\begin{bmatrix} 1 & 0 \\ 1 & 11 \end{bmatrix}}_{2 \times 2}$$

N.B

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}_{2 \times 2}$$

$$B = \begin{bmatrix} 4 & 5 \end{bmatrix}_{1 \times 2}$$

$\Rightarrow$
$$\left.\begin{array}{l} A + B : \text{Does not exist} \\ A - B : \text{Does not exist} \\ B - A : \text{Does not exist} \end{array}\right\} \text{cause } 2 \times 2 \neq 1 \times 2$$

Scalar multiplication of matrix

- Multiply a matrix $A\,(m \times n)$ by a scalar K:

$$A = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}_{2\times 2} \Rightarrow 3A = \begin{bmatrix} 6 & 3 \\ 9 & 12 \end{bmatrix}_{2\times 2}$$

Properties

Let A, B be matrices and K, P scalars then the following properties:

1. $K(A+B) = KA + KB$
2. $(K+P)A = KA + PA$
3. $K(pA) = (Kp)A$
4. $1 \times A = A$

Multiplication of matrices

- To multiply two matrices A and B:
  - The numbre of Columns in A should be equal the mubre of rows in B
  - $A\,(n \times r)$; $B\,(r \times m)$.
  - Then $C = A.B$ is a matrix with $(n \times m)$ dimension

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}_{2\times 3} \qquad \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix}_{3\times 2}$$

if $C = A.B$ then

$$C = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}_{2\times 2}$$

Where:

C_{11} Σ[(1st row in A) (1st column in B)]

C_{12} Σ[(1st row in A) (2nd column in B)]

C_{21} Σ[(2nd row in A) (1st column in B)]

C_{22} Σ[(2nd row in A) (2nd column in B)]

Examples: $A \begin{bmatrix} 1 & 2 \\ 10 & 1 \end{bmatrix}_{2\times 2}$ $B \begin{bmatrix} 2 \\ 1 \end{bmatrix}_{2\times 1}$ find AB

$$AB = C \begin{bmatrix} c_{11} \\ c_{21} \end{bmatrix}_{2\times 1} = C \begin{bmatrix} 2+2 \\ 20+1 \end{bmatrix}_{2\times 1} = \begin{bmatrix} 4 \\ 21 \end{bmatrix}_{2\times 1}$$

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 4 & 6 \end{bmatrix}_{2\times 3} \qquad B = \begin{bmatrix} 2 & 3 & 4 \\ -1 & 0 & 5 \\ 1 & 2 & 3 \end{bmatrix}_{3\times 3}$$

if $C = A.B$, then:

$$C = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \end{bmatrix}_{2\times 3}$$

$$C = \begin{bmatrix} 2+0+3 & 3+0+6 & 4+0+9 \\ 4+(-4)+6 & 6+0+12 & 8+20+18 \end{bmatrix}_{2\times 3}$$

$$C = \begin{bmatrix} 5 & 9 & 13 \\ 6 & 18 & 46 \end{bmatrix}_{2\times 3}$$

N.B: In this case we can't multiply B.A cause

$$3\times \underbrace{3 \cdot 2}_{\neq}\times 3$$

Example: Multiplication by O

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}_{2\times 3} \qquad O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}_{3\times 2}$$

$$A \times O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}_{2\times 2}$$

Properties

A, B, C 3 matrices and r, s scalars

1. $A(rB + sC) = r(AB) + s(AC)$
2. $(B + C)A = BA + CA$
3. $A(BC) = (AB)C$

Example:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}_{2\times 2} \qquad B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}_{2\times 2}$$

Find AB and BA then compare them.

$$AB = \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix}_{2\times 2} \qquad BA = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}_{2\times 2}$$

$\Rightarrow BA \neq AB$

→ The transpose

We get the transpose of a matrix $A\ (m \times n)$ by making the row a column and the column a row $\Rightarrow A^t\ (n \times m)$

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix}_{2\times4} \Longrightarrow A^t = \begin{bmatrix} 1 & 5 \\ 2 & 6 \\ 3 & 7 \\ 4 & 8 \end{bmatrix}$$

↳ Properties

Let $A\ (m \times n)$; $B\ (n \times p)$ matrices and r, s scalars.

1. $(A^t)^t = A$
2. $(AB)^t = B^t \times A^t$ نعكس ترتيب ال matrix
3. $(rA + sB)^t = rA^t + sB^t$
4. $(A + B)^t = A^t + B^t$

→ Symetric and skew symetric matrix

An $(n \times n)$ matrix A:

- if $A = A^t \Rightarrow A$ is symetric
- if $A = -A^t \Rightarrow A$ is skew symetric

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 5 & -3 \\ 3 & -3 & 7 \end{bmatrix}_{3\times3} \qquad A^t = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 5 & -3 \\ 3 & -3 & 7 \end{bmatrix}_{3\times3}$$

$A = A^t$

A is symetric

$$B = \begin{bmatrix} 0 & 1 & 3 \\ -1 & 0 & 2 \\ -3 & -2 & 0 \end{bmatrix} \qquad B^t = \begin{bmatrix} 0 & -1 & -3 \\ 1 & 0 & -2 \\ 3 & 2 & 0 \end{bmatrix}$$

$B = -B^t$

B is skew symetric

→ Identity and Inverse

- I_n ; $(n \times n)$ matrix ;

$$\underset{I_1}{\begin{bmatrix} 1 \end{bmatrix}_{1\times1}} \quad \underset{I_2}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_{2\times2}} \quad \underset{I_3}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3\times3}} \quad \underset{I_4}{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}_{4\times4}}$$

- $I_3 \, A_{3\times6} = A_{3\times6}$
- $A_{4\times3} \, I_3 = A_{4\times3}$
- $A_{3\times6} \, I_3$: does not exist.

→ Inverse of a matrix

- A square matrix $A\,(n\times n)$ is said to have an inverse A^{-1} if and only if :
  $A . A^{-1} = A^{-1} . A = I_n$

Then A is invertible and A^{-1} is the inverse of A.

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}_{2\times2} \quad A^{-1} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}_{2\times2} \quad A.A^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

$\Rightarrow$ $A . A^{-1} = I_2$

A is invertible

A^{-1} is the inverse of A.

→ Finding the inverse of a matrix

- $A\ (n \times n)$ matrix
- To find A^{-1} if it exist :

1. Form the augmented matrix $(n \times 2n)$:

$$[A \mid I_n]_{n \times 2n}$$

2. Do row operation until we obtain we obtain $(n \times 2n)$ matrix of the form.

$$[I_n \mid B]_{n \times 2n}$$

3. When this has been done then :
   B is the inverse A^{-1} of A.
   A is invertible

Example :

I)

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 0 & 2 \\ 3 & 1 & -1 \end{bmatrix}$$

Find A^{-1} if exist

Augmented matrix :

هذه تبقى 1

$$[A \mid I_3] = \left[\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 1 & 0 & 2 & 0 & 1 & 0 \\ 3 & 1 & -1 & 0 & 0 & 1 \end{array}\right]_{3 \times 6}$$

يجب أن أحولها إلى 0 إستناداً على الـ diagonal element

$R_2 - R_1 = 1 - 1 = 0$

- يجب تحويل A إلى I_3
- بالتالي تحويل كل element في A إلى الـ element المناسب في I_3 (1 or 0)

يتم التحويل عن طريق operations و نستمر في التحويل على الـ diagonal elements

Row operations:

$$\begin{array}{c}\\ 0 \leftarrow \\ \\ \end{array}\left[\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 1 & 0 & 2 & 0 & 1 & 0 \\ 3 & 1 & -1 & 0 & 0 & 1 \end{array}\right]$$

$R_2 - R_1 \rightarrow R_2$

$(1) - (1)$

نحل كل ال row R_2

$$\begin{array}{c}\\ \\ 0 \leftarrow \end{array}\left[\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & -2 & 0 & -1 & 1 & 0 \\ 3 & 1 & -1 & 0 & 0 & 1 \end{array}\right]$$

$R_3 - 3R_1 \rightarrow R_3$

$3 - 3(1)$

$3 + x = 0$

$x = -3$

$3 - 3$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & -2 & 0 & -1 & 1 & 0 \\ 0 & \underset{\searrow 0}{-5} & -7 & -3 & 0 & 1 \end{array}\right]$$

$R_3 - \frac{5}{2} R_2 \rightarrow R_3$

$-5 + x(-2) = 0$

$-2x = 5$

$x = \frac{5}{-2}$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & -2 & 0 & -1 & 1 & 0 \\ 0 & 0 & \underset{\searrow 1}{-7} & -\frac{1}{2} & -\frac{5}{2} & 1 \end{array}\right]$$

$R_3 / -7 \rightarrow R_3$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & \overset{\nearrow 0}{2} & 1 & 0 & 0 \\ 0 & -2 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & \frac{1}{14} & \frac{5}{14} & -\frac{1}{7} \end{array}\right]$$

$R_1 - 2R_3 \rightarrow R_1$

$$\left[\begin{array}{ccc|ccc} 1 & & 0 & \frac{6}{7} & -\frac{5}{7} & \frac{2}{7} \\ 0 & -2 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & \frac{1}{14} & \frac{5}{14} & -\frac{1}{7} \end{array}\right]$$

$R_1 + R_2 \rightarrow R_1$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{7} & \frac{2}{7} & \frac{2}{7} \\ 0 & -2 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & \frac{1}{14} & \frac{5}{14} & -\frac{1}{7} \end{array}\right]$$

$R_2/2 \to R_2$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{7} & \frac{2}{7} & \frac{2}{7} \\ 0 & 1 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & \frac{1}{14} & \frac{5}{14} & -\frac{1}{7} \end{array}\right]$$

$\underbrace{\quad\quad}_{I_3}$ $\underbrace{\quad\quad}_{A^{-1}}$

II)

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 0 & 2 \\ 2 & 2 & 4 \end{bmatrix}$$

Find A^{-1} if exist

Augmented Matrix:

$$[A | I_3] = \left[\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 1 & 0 & 2 & 0 & 1 & 0 \\ 2 & 2 & 4 & 0 & 0 & 1 \end{array}\right]$$

Row operations:

$R_2 - R_1 \to R_2$
$$\left[\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & -2 & 0 & -1 & 1 & 0 \\ 2 & 2 & 4 & 0 & 0 & 1 \end{array}\right]$$

$R_3 - 2R_1 \to R_3$
$$\left[\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & -2 & 0 & -1 & 1 & 0 \\ 0 & -2 & 0 & -2 & 0 & 1 \end{array}\right]$$

$R_3 - R_2 \to R_3$
$$\left[\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & -2 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & -1 & 1 \end{array}\right]$$

$\Rightarrow A^{-1}$ does not exist

"إذا واحد من (diagonal element) يساوي 0 إذاً
لا يوجد inverse"

Theoreme : Inverse of transpose and products

Let A, B and A_i for $i = 1 \dots K$ be $n \times m$ matrices and I_n the identity matrix

1. If A is an invertible matrix then: $(A^t)^{-1} = (A^{-1})^t$

2. If A and B are invertible matrices then AB is invertible and $(AB)^{-1} = B^{-1} A^{-1}$

3. If $A_1, A_2, \dots A_K$ are invertible matrices then the product $A_1 A_2 \dots A_K$ is inv $(A_1 A_2 \dots A_K)^{-1} = A_K^{-1} A_{K-1}^{-1} \dots A_1^{-1}$

4. I_n is invertible : $I_n^{-1} = I_n$

5. If A is invertible then: $(A^{-1})^{-1} = A$

6. If A is invertible then: $(A^K)^{-1} = (A^{-1})^K$
   $A^2 = A \times A$; $A^3 = A \times A \times A$ or $A^2 \times A$.